for all integers $k \geq 2$. Then, a level-k fuzzy set A is defined by

$$\mu_A: \tilde{\mathcal{P}}^{k-1}(U) \rightarrow [0, 1]. \qquad (2.47)$$

2.2.2 Further Operations on Fuzzy Sets

As we have mentioned, the standard operations, that is, negation, min, and max operations in Eqs. (2.24), (2.25), and (2.27), respectively, are not the only possible generalization of the crisp set complement, intersection, and union operations. This raises a question concerning the requirements, specifications, and properties of other functions that can be viewed as a generalization of the crisp set operations. We shall first discuss several different classes of functions for each of the above three standard set operators. These functions will possess appropriate properties. For each operation, the corresponding functions can be divided into two categories. One is nonparametric functions such as Eqs. (2.24), (2.25), and (2.27), and the other is parametric functions in which parameters are used to adjust the "strength" of the corresponding operations. Based on these functions, we will introduce other kinds of fuzzy set operations. Let us consider the fuzzy complement first.

A complement of a fuzzy set A, denoted as $\bar{A}$, is specified by a function

$$c: [0, 1] \rightarrow [0, 1], \qquad (2.48)$$

such that

$$\mu_{\bar{A}}(x) = c(\mu_A(x)), \qquad (2.49)$$

where the function $c(\cdot)$ satisfies the following conditions:

c1. *Boundary conditions:* $c(0) = 1$ and $c(1) = 0$.
c2. *Monotonic property:* For any $x_1, x_2 \in U$, if $\mu_A(x_1) < \mu_A(x_2)$, then $c(\mu_A(x_1)) \geq c(\mu_A(x_2))$; that is, $c(\cdot)$ is monotonic nonincreasing.
c3. *Continuity:* $c(\cdot)$ is a continuous function.
c4. *Involution:* $c(\cdot)$ is involutive, which means that

$$c(c(\mu_A(x))) = \mu_A(x), \qquad \forall x \in U. \qquad (2.50)$$

Based on the above conditions, typical examples of nonparametric and parametric fuzzy complements are

1. *Negation complement:* The complement of A using this operation is denoted as $\bar{A}$ and is defined as in Eq. (2.24), that is,

$$\mu_{\bar{A}}(x) = c(\mu_A(x)) \triangleq 1 - \mu_A(x) \qquad \forall x \in U. \qquad (2.51)$$

2. λ *Complement (Sugeno class):* This complement is denoted as $\bar{A}^{\lambda}$ and is defined by

$$\mu_{\bar{A}^{\lambda}}(x) = c(\mu_A(x)) \triangleq \frac{1 - \mu_A(x)}{1 + \lambda\mu_A(x)}, \qquad -1 < \lambda < \infty. \qquad (2.52)$$

λ is a parameter that gives the degree of complementation. When $\lambda = 0$, the function becomes $c(\mu_A(x)) = 1 - \mu_A(x)$, the standard fuzzy complement, and as λ approaches -1, $\bar{A}^{\lambda}$ approaches the universal set U. When λ approaches infinity, $\bar{A}^{\lambda}$

approaches the empty set. It is noted that the double-negation law [Eq. (2.34)] and DeMorgan's laws [Eqs. (2.35) and (2.36)] hold for the λ-complement.

3. *w Complement (Yager class):* This complement is denoted as $\bar{A}^w$ and is defined by

$$\mu_{\bar{A}^w}(x) = c(\mu_A(x)) \triangleq (1 - \mu_A^w(x))^{1/w}, \qquad 0 < w < \infty. \tag{2.53}$$

Again, the parameter w adjusts the degree of complementation. When $w = 1$, the w-complement function becomes the standard fuzzy complement of $c(\mu_A(x)) = 1 - \mu_A(x)$.

The *equilibrium* of a fuzzy complement c is defined as any value a for which $c(a) = a$. For example, the equilibrium of the standard complement operation is 0.5, which is the solution of the equation $1 - a = a$. An important property shared by all fuzzy complements is that every fuzzy complement has at most one equilibrium due to the monotonic nonincreasing nature of fuzzy complements.

Next, let us discuss the intersection and union operations of fuzzy sets, which are often referred to as *triangular norms (t-norms)* and *triangular conorms (t-conorms)*, respectively [Dubois and Prade, 1980, 1985b]. t-norms are two-parameter functions of the form

$$t\colon [0, 1] \times [0, 1] \rightarrow [0, 1], \tag{2.54}$$

such that

$$\mu_{A \cap B}(x) = t[\mu_A(x), \mu_B(x)], \tag{2.55}$$

where the function $t(\cdot, \cdot)$ satisfies the following conditions:

t1. *Boundary conditions:* $t(0, 0) = 0$; $t(\mu_A(x), 1) = t(1, \mu_A(x)) = \mu_A(x)$.
t2. *Commutativity:* $t(\mu_A(x), \mu_B(x)) = t(\mu_B(x), \mu_A(x))$.
t3. *Monotonicity:* If $\mu_A(x) \leq \mu_C(x)$ and $\mu_B(x) \leq \mu_D(x)$, then $t(\mu_A(x), \mu_B(x)) \leq t(\mu_C(x), \mu_D(x))$.
t4. *Associativity:* $t(\mu_A(x), t(\mu_B(x), \mu_C(x))) = t(t(\mu_A(x), \mu_B(x)), \mu_C(x))$.

The t-norms and t-conorms are also used to define other operations. Typical nonparametric t-norms are [to simplify the notation, we use $a \equiv \mu_A(x)$ and $b = \mu_B(x)$ for the remainder of this section only]:

1. Intersection: $a \wedge b = \min(a, b)$. (2.56)
2. Algebraic product: $a \cdot b = ab$. (2.57)
3. Bounded product: $a \odot b = \max(0, a + b - 1)$. (2.58)
4. Drastic product: $$a \mathbin{\hat{\cdot}} b = \begin{cases} a, & b = 1, \\ b, & a = 1, \\ 0, & a, b < 1. \end{cases} \tag{2.59}$$

One representative parametric t-norm is the *Yager intersection* which is defined by the function

$$t_w(a, b) = 1 - \min[1, ((1 - a)^w + (1 - b)^w)^{1/w}], \tag{2.60}$$

where $w \in (0, \infty)$. For $w = 1$, the Yager intersection becomes the bounded product of Eq. (2.58). It can be shown that when $w \rightarrow \infty$, $t_w(a, b) = \min(a, b)$, and when $w \rightarrow 0$, $t_w(a, b)$ becomes the drastic product [Klir and Folger, 1988]. That is, the Yager intersection becomes the min operator when $w \rightarrow \infty$. It is observed that the membership grade increases as w

increases. Hence, the parameter w can be interpreted as the degree of strength of intersection performed. Some other classes of parametric t-norms are shown in Table 2.3.

t-conorms (also called s-norms) are two-parameter functions of the form

$$s: [0, 1] \times [0, 1] \rightarrow [0, 1]. \tag{2.61}$$

such that

$$\mu_{A \cup B}(x) = s[\mu_A(x), \mu_B(x)]. \tag{2.62}$$

where the function $s(\cdot, \cdot)$ satisfies the following conditions:

s1. *Boundary conditions:* $s(1, 1) = 1$, $s(\mu_A(x), 0) = s(0, \mu_A(x)) = \mu_A(x)$.
s2. *Commutativity:* $s(\mu_A(x), \mu_B(x)) = s(\mu_B(x), \mu_A(x))$.
s3. *Monotonicity:* If $\mu_A(x) \leq \mu_C(x)$ and $\mu_B(x) \leq \mu_D(x)$, then $s(\mu_A(x), \mu_B(x)) \leq s(\mu_C(x), \mu_D(x))$.
s4. *Associativity:* $s(\mu_A(x), s(\mu_B(x), \mu_C(x))) = s(s(\mu_A(x), \mu_B(x)), \mu_C(x))$.

Based on the above conditions, typical nonparametric t-conorms are

1. Union: $a \vee b = \max(a, b)$. (2.63)
2. Algebraic sum: $a \hat{+} b = a + b - ab$. (2.64)

TABLE 2.3 Some Parameterized t-Norms (Fuzzy Intersections) and t Conorms (Fuzzy Unions), where $a \equiv \mu_A(x)$ and $b \equiv \mu_B(x)$.

References	t norms (Fuzzy Intersections)	t conorms (Fuzzy Unions)	Range
Schweizer and Sklar [1961]	$\max\{0, a^{-r} + b^{-r} - 1\}^{-\frac{1}{r}}$	$1 - \max\{0, (1-a)^{-r} + (1-b)^{-r} - 1\}^{-\frac{1}{r}}$	$r \in (-\infty, \infty)$
Hamacher [1978]	$\dfrac{ab}{\gamma + (1-\gamma)(a+b-ab)}$	$\dfrac{a+b-(2-\gamma)ab}{1-(1-\gamma)ab}$	$\gamma \in (0, \infty)$
Frank [1979]	$\log_s\left[1 + \dfrac{(s^a-1)(s^b-1)}{s-1}\right]$	$1 - \log_s\left[1 + \dfrac{(s^{1-a}-1)(s^{1-b}-1)}{s-1}\right]$	$s \in (0, \infty)$
Yager [1980]	$1 - \min\{1, ((1-a)^w + (1-b)^w)^{\frac{1}{w}}\}$	$\min\{1, (a^w + b^w)^{\frac{1}{w}}\}$	$w \in (0, \infty)$
Dubois and Prade [1980]	$\dfrac{ab}{\max\{a, b, \alpha\}}$	$\dfrac{a+b-ab-\min\{a, b, 1-\alpha\}}{\max\{1-a, 1-b, \alpha\}}$	$\alpha \in (0, 1)$
Dombi [1982]	$1/1 + \left[\left(\frac{1}{a}-1\right)^{\lambda} + \left(\frac{1}{b}-1\right)^{\lambda}\right]^{\frac{1}{\lambda}}$	$1/1 + \left[\left(\frac{1}{a}-1\right)^{-\lambda} + \left(\frac{1}{b}-1\right)^{-\lambda}\right]^{-\frac{1}{\lambda}}$	$\lambda \in (0, \infty)$
Werners [1988]	$\beta \min\{a, b\} + \dfrac{(1-\beta)(a+b)}{2}$	$\beta \max\{a, b\} + \dfrac{(1-\beta)(a+b)}{2}$	$\beta \in [0, 1]$
Zimmermann and Zysno [1980]	$(ab)^{(1-\gamma)}[1-(1-a)(1-b)]^{\gamma}$	γ parameter indicates compensation between intersection and union	$\gamma \in [0, 1]$

3. Bounded sum: $a \oplus b = \min(1, a + b)$. (2.65)

4. Drastic sum: $a \dot{\vee} b = \begin{cases} a, & b = 0, \\ b, & a = 0. \\ 1, & a, b > 0. \end{cases}$ (2.66)

5. Disjoint sum: $a \triangle b = \max\{\min(a, 1 - b), \min(1 - a, b)\}$ (2.67)

One typical parametric t-conorm is the *Yager union* which is defined by the function

$$s_w(a, b) = \min[1, (a^w + b^w)^{1/w}], \tag{2.68}$$

where $w \in (0, \infty)$. For $w = 1$, the Yager union becomes the bounded sum of Eq. (2.65). We can show that when $w \to \infty$, $s_w(a, b) = \max(a, b)$, and when $w \to 0$, $s_w(a, b)$ becomes the drastic sum [Klir and Folger, 1988]. It is observed that the membership grade decreases as w increases. Some other classes of parametric t-conorms are also shown in Table 2.3.

The relations among various t-norms (fuzzy intersections) and t-conorms (fuzzy unions) are characterized by the following theorem.

Theorem 2.2

Let A and B be fuzzy sets in the universe of discourse U. The t-norms [Eqs. (2.56)–(2.59)] are bounded by the inequalities

$$t_{dp}(a, b) = t_{min}(a, b) \leq t(a, b) \leq t_{max}(a, b) = \min(a, b), \tag{2.69}$$

where $t_{dp}(a, b)$ is the drastic product in Eq. (2.59). Similarly, the t-conorms [Eqs. (2.63)–(2.67)] are bounded by the inequalities

$$\max(a, b) = s_{min}(a, b) \leq s(a, b) \leq s_{max}(a, b) = s_{ds}(a, b), \tag{2.70}$$

where $s_{ds}(a, b)$ is the drastic sum in Eq. (2.66).

Proof: Since the proof of Eq. (2.70) is similar to that of Eq. (2.69), we shall prove only Eq. (2.69). Using the boundary condition of the t-norm, we have $t(a, 1) = a$ and $t(1, b) = b$. Then by the monotonicity condition of the t-norm, we obtain

$$t(a, b) \leq t(a, 1) = a \qquad \text{and} \qquad t(a, b) \leq t(1, b) = b.$$

Hence, we conclude that

$$t(a, b) \leq \min(a, b),$$

which is the second inequality in Eq. (2.69). For the first inequality in Eq. (2.69), when $b = 1$, $t(a, b) = a$, and when $a = 1$, $t(a, b) = b$ (boundary conditions). Hence, the first inequality holds when $a = 1$ or $b = 1$. Since $t(a, b) \in [0, 1]$, it follows from the second inequality in Eq. (2.69) that $t(a, 0) = t(0, b) = 0$. By the monotonicity condition, we have

$$t(a, b) \geq t(0, b) = t(a, 0) = 0,$$

which completes the proof of the first inequality in Eq. (2.69).

Hence, the standard min and max operations are, respectively, the upper bound of t-norms (the weakest intersection) and the lower bound of t-conorms (the strongest union). As mentioned before, the Yager intersection [Eq. (2.60)] and the Yager union [Eq. (2.68)] become the standard min and max operations, respectively, as $w \to \infty$, and become the t_{min} and s_{max} operations, respectively, as $w \to 0$. Hence, the Yager class of fuzzy intersections and unions